Fourier transform
with *Mathematica*

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```
Print["Revision ", IntegerPart[Date[]]]
```

```
Revision {2013, 8, 29, 7, 11, 55}
```

```
Print["This system is:"]
{"ProductIDName", "ProductVersion"} /. $ProductInformation
ReadList["!ver", String][[2]]
{$MachineType, $ProcessorType, $ByteOrdering, $SystemCharacterEncoding}
```

```
This system is:

{Mathematica, 5.2 for Microsoft Windows (June 20, 2005)}

Windows 98 [versione 4.10.1998]

{PC, x86, -1, WindowsANSI}
```

5.2 Trasformate

5.2.0 Il concetto di trasformata *("From hell to heaven and back again...")*

```
{N[Pi, 11], RealDigits[Pi, Pi, 2], RealDigits[3 Pi^2, Pi, 3],
 RealDigits[2 Pi^-4, Pi, 3], A = RealDigits[10., Pi, 13], FromDigits[A, Pi] // Expand}
```

$$\left\{3.1415926536, \{\{0, 3\}, 2\}, \{\{2, 3, 0\}, 3\}, \{\{1, 3, 0\}, -3\}, \{\{1, 0, 0, 0, 1, 0, 2, 2, 1, 2, 2, 2, 2\}, 3\}, \frac{2}{\pi^{10}} + \frac{2}{\pi^9} + \frac{2}{\pi^8} + \frac{2}{\pi^7} + \frac{1}{\pi^6} + \frac{2}{\pi^5} + \frac{2}{\pi^4} + \frac{1}{\pi^2} + \pi^2\right\}$$

5.2.1 La trasformata di Fourier

- **The Fourier transform of a function $f(t)$ is by default defined to be $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$.**

- **The inverse Fourier transform of a function $F(\omega)$ is by default defined as $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega t}\, d\omega$.**

- *Campana razionale (Un caso semplice)*

```
campanarazionale[t_] = n/(π (1 + n^2 t^2))
Fcampanarazionale[ω_] =
 FourierTransform[campanarazionale[t], t, ω, Assumptions → {n > 0}]
% // PiecewiseExpand
Fcampanarazionale2[ω_] = E^(-Abs[ω]/n)/Sqrt[2 π]
Fcampanarazionale[ω] == Fcampanarazionale2[ω];
{FullSimplify[%, ω > 0], FullSimplify[%, ω < 0]}
Limit[Fcampanarazionale[ω], ω → 0];
{%,
 Fcampanarazionale[0], Fcampanarazionale2[0] == %}
InverseFourierTransform[Fcampanarazionale[ω], ω, t] // Together
InverseFourierTransform[Fcampanarazionale2[ω], ω, t]
```

$$\frac{n}{\pi\left(1+n^2 t^2\right)}$$

$$\frac{e^{-\frac{\omega}{n}}\left(e^{\frac{2\omega}{n}}\,\mathrm{UnitStep}[-\omega]+\mathrm{UnitStep}[\omega]\right)}{\sqrt{2\pi}}$$

$$\begin{cases} \frac{e^{-\frac{\omega}{n}}}{\sqrt{2\pi}} & \omega > 0 \\ \frac{e^{\frac{\omega}{n}}}{\sqrt{2\pi}} & \omega < 0 \\ \frac{e^{-\frac{\omega}{n}}\left(1+e^{\frac{2\omega}{n}}\right)}{\sqrt{2\pi}} & \mathrm{True} \end{cases}$$

$$\frac{e^{-\frac{\mathrm{Abs}[\omega]}{n}}}{\sqrt{2\pi}}$$

{True, True}

$$\left\{\frac{1}{\sqrt{2\pi}}, \sqrt{\frac{2}{\pi}}, \mathrm{True}\right\}$$

$$\frac{n}{\pi\left(1+n^2 t^2\right)}$$

$$\frac{n}{\pi\left(1+n^2 t^2\right)}$$

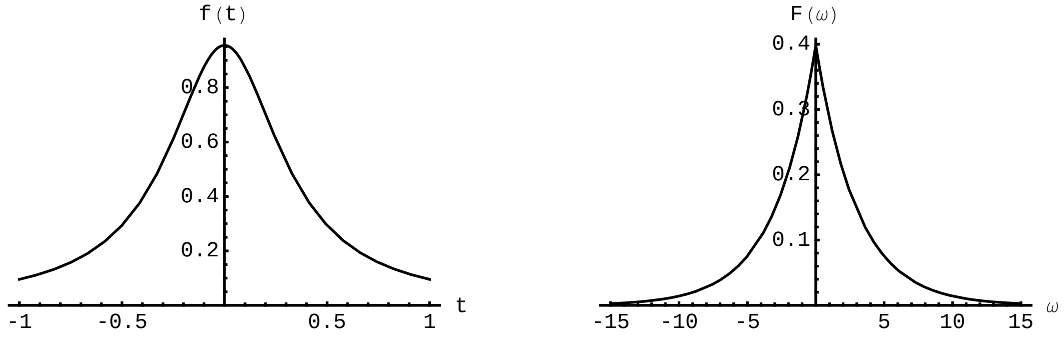
```
n = 3;
campanarazionale[t_] = n / (π (1 + n^2 t^2))
Fcampanarazionale[ω_] = FourierTransform[campanarazionale[t], t, ω]
{Limit[Fcampanarazionale[ω], ω → 0],
 Fcampanarazionale[0]}
```

$$\frac{3}{\pi\left(1+9\,t^2\right)}$$

$$\frac{e^{-\frac{\text{Abs}[\omega]}{3}}}{\sqrt{2\,\pi}}$$

$$\left\{\frac{1}{\sqrt{2\,\pi}},\ \frac{1}{\sqrt{2\,\pi}}\right\}$$

```
p1 = Plot[campanarazionale[t], {t, -1, +1}, PlotRange → {0, Automatic},
   AxesLabel → {"t", "f(t)"}, DisplayFunction → Identity];
p2 = Plot[Fcampanarazionale[ω], {ω, -15, +15}, PlotRange → {0, Automatic},
   AxesLabel → {"ω", "F(ω)"}, DisplayFunction → Identity];
Show[GraphicsArray[{p1, p2}]];
```



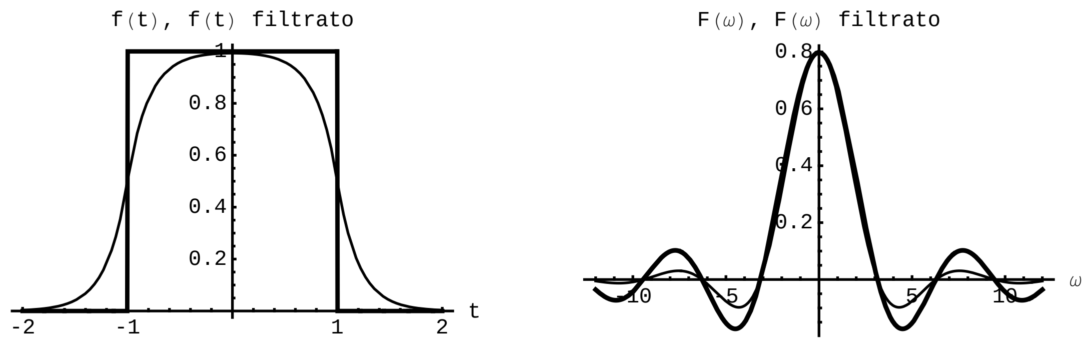
- ***Porta (Effetto di un filtro passa-basso)***

```
porta[t_] = UnitStep[t + a] - UnitStep[t - a]
Fporta[ω_] = FourierTransform[porta[t], t, ω] // ExpToTrig // FullSimplify
filtro[ω_] := 1 / (1 + .04 ω^2); Fportatrunc[ω_] := Fporta[ω] filtro[ω];
portatrunc[t_] = InverseFourierTransform[Fportatrunc[ω], ω, t];
```

-UnitStep[-a + t] + UnitStep[a + t]

$$\frac{\sqrt{\frac{2}{\pi}}\ \text{Sin}[a\,\omega]}{\omega}$$

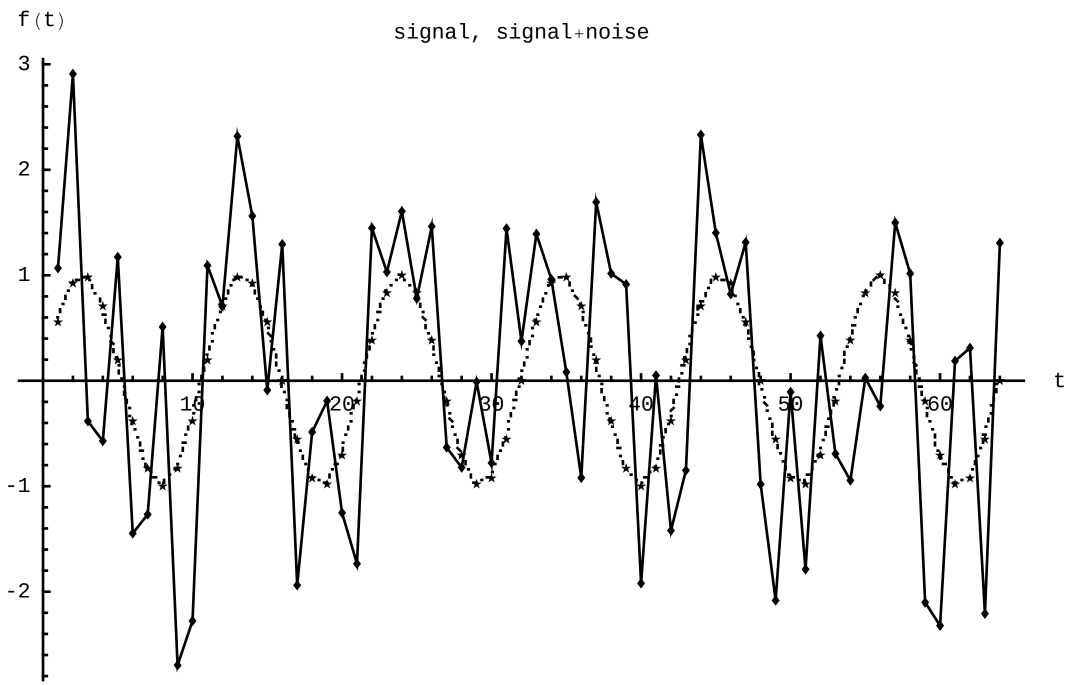
```
a = 1; p1 = Plot[{porta[t], portatrunc[t]},
  {t, -2, +2}, PlotRange → All, PlotStyle → {{Thickness[0.01]}, {}},
  AxesLabel → {"t", "f(t), f(t) filtrato"}, DisplayFunction → Identity];
p2 = Plot[{Fporta[ω], Fportatrunc[ω]}, {ω, -12, 12},
   PlotRange → All, PlotStyle → {{Thickness[0.01]}, {}},
   AxesLabel → {"ω", "F(ω), F(ω) filtrato"}, DisplayFunction → Identity];
Show[GraphicsArray[{p1, p2}]];
```

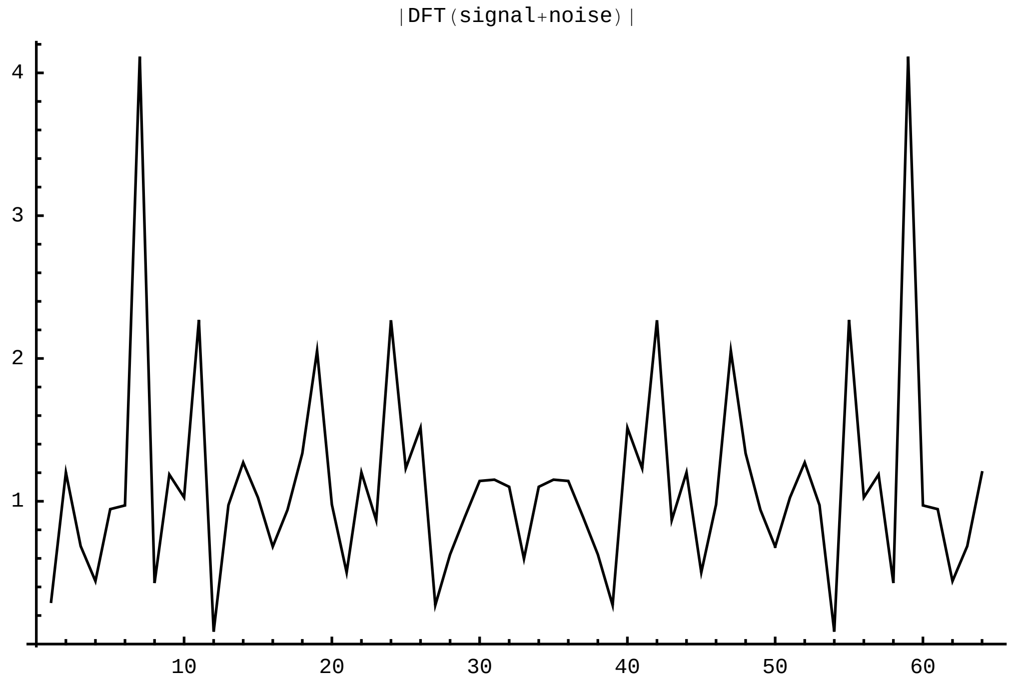


- ***Analisi dati (Dov'è la sinusoide?)***

```
<< Graphics`MultipleListPlot`

freq=12/128;noise=4;nsample=64;
{freq,1/freq,freq nsample,nsample(1-freq)}//N
signal=Table[N[Sin[2 Pi freq n]],{n,nsample}];
%//Short
data=Table[signal[[n]]+noise(Random[]-1/2),{n,nsample}];
%//Short
Fourier[data]//Chop;
Short[%,5]
Fdata = Abs[%%];
%//Short
Position[Fdata,Max[Fdata]]
Plus[%,-1]/nsample//N
MultipleListPlot[{data,signal},PlotJoined  -> {True, True},PlotLabel→"signal,  signal+noise
ListPlot[Fdata, PlotJoined→True,PlotLabel→"|DFT(signal+noise)|",PlotRange →All];
```

```
{0.09375, 10.6667, 6., 58.}

{0.55557, 0.92388, <<61>>, 0.}

{1.06779, <<62>>, 1.30614}

{0.296721, -1.10087 + 0.482254 i, 0.65727 + 0.197185 i,
 <<59>>, 0.65727 - 0.197185 i, -1.10087 - 0.482254 i}

{0.296721, <<62>>, 1.20186}

{{7}, {59}}

{{0.09375}, {0.90625}}
```





- ***Denti di sega (Serie di Fourier per una funzione periodica)***

```
<< Calculus`FourierTransform`
```

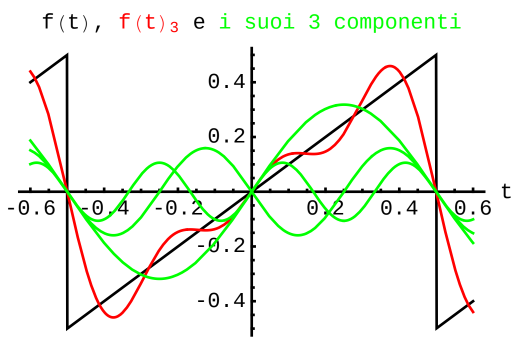
```
sawtooth[t_] = t - Round[t]
sawtooth3[t_] = FourierTrigSeries[t, t, 3]
sawtooth3N[t_] = NFourierTrigSeries[sawtooth[t], t, 3] // Chop
sawtooth10N[t_] = NFourierTrigSeries[sawtooth[t], t, 10] // Chop;
```

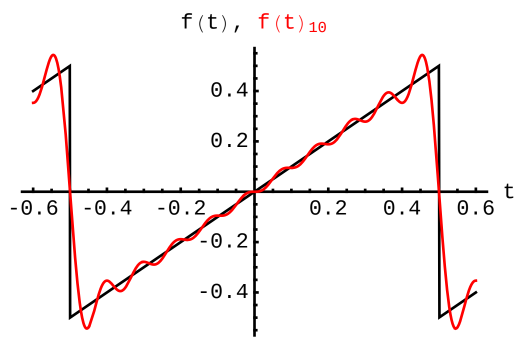
t - Round[t]

$$\frac{\text{Sin}[2\pi t]}{\pi} - \frac{\text{Sin}[4\pi t]}{2\pi} + \frac{\text{Sin}[6\pi t]}{3\pi}$$

0.31831 Sin[2 π t] - 0.159155 Sin[4 π t] + 0.106103 Sin[6 π t]

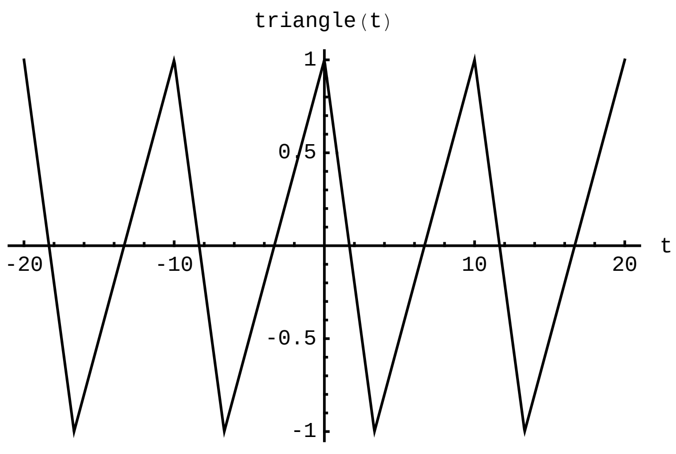
```
p1 = Plot[Evaluate[{sawtooth[t], sawtooth3N[t], Sequence @@ sawtooth3N[t]}],
  {t, -.6, .6}, PlotStyle → {{}, {Hue[0]}, {Hue[1/3]}, {Hue[1/3]}, {Hue[1/3]}},
  AxesLabel → {"t", "f(t), f(t)3 e i suoi 3 componenti"}, DisplayFunction → Identity];
p2 = Plot[{sawtooth[t], sawtooth10N[t]}, {t, -.6, .6},
  PlotStyle → {{}, {Hue[0]}}, AxesLabel → {"t", "f(t), f(t)10"},
  DisplayFunction → Identity]; Show[GraphicsArray[{p1, p2}]];
```





- ***Triangle***

```
T = 10; n = 3; a = 1; b = T / n;
core[t_] := { a + (-2 a / b) t                               0 ≤ t < T / n
            { (2 a / (T - b)) t - (2 a / (T - b)) (T / n) - a   T / n ≤ t < T
triangle[t_] := core[Mod [t, T]]
Plot[Evaluate[triangle[t] ], {t, -2 T, 2 T}, AxesLabel → {"t", "triangle(t)"}];
```




```
<< Calculus`FourierTransform`
```

```
triangle2[t_] =
 FourierTrigSeries[triangle[t], t, 2, FourierParameters -> {0, 1 / T}] // Expand
triangle2N[t_] = triangle2[t] // N // Simplify
```

$$\frac{27\,\text{Cos}\left[\frac{\pi t}{5}\right]}{4\,\pi^2}+\frac{27\,\text{Cos}\left[\frac{2\pi t}{5}\right]}{16\,\pi^2}-\frac{9\sqrt{3}\,\text{Sin}\left[\frac{\pi t}{5}\right]}{4\,\pi^2}+\frac{9\sqrt{3}\,\text{Sin}\left[\frac{2\pi t}{5}\right]}{16\,\pi^2}$$

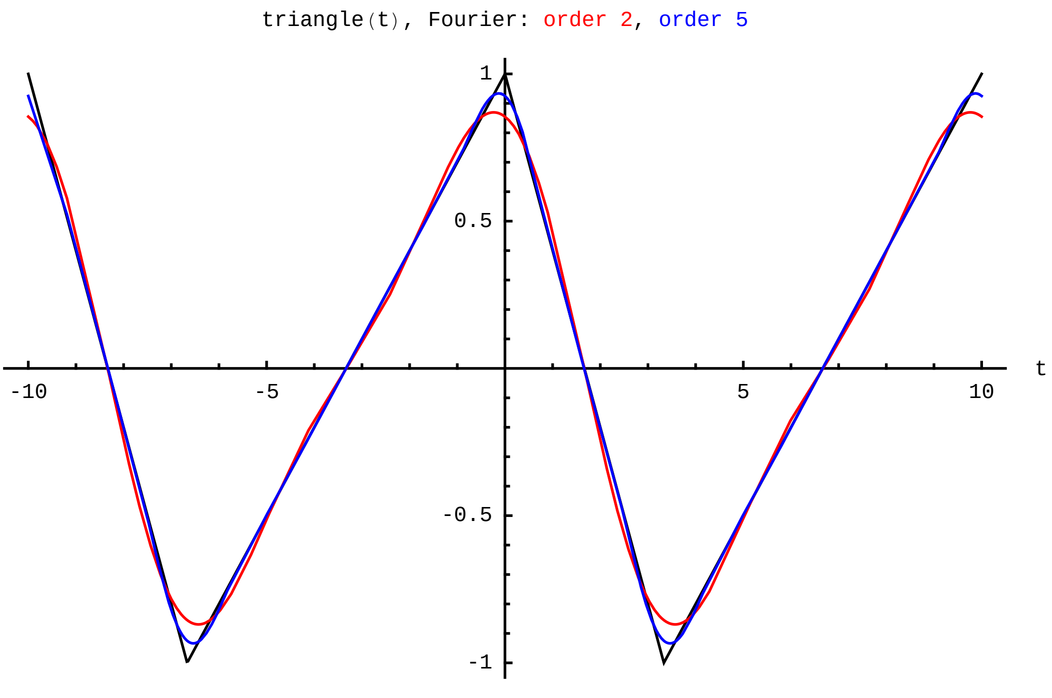
```
0.683918 Cos[0.628319 t] + 0.170979 Cos[1.25664 t] -
 0.39486 Sin[0.628319 t] + 0.0987151 Sin[1.25664 t]
```

```
NFourierTrigSeries[triangle[t], t, 5, FourierParameters -> {0, 1 / T}]
triangle5N[t_] = Chop[% // N // Simplify, 10^-7]
```

$$\frac{1}{\sqrt{10}}\left(1.40433\times10^{-16}+2.16274\,\text{Cos}\left[\frac{\pi t}{5}\right]+0.540685\,\text{Cos}\left[\frac{2\pi t}{5}\right]-1.75542\times10^{-17}\,\text{Cos}\left[\frac{3\pi t}{5}\right]+0.135171\,\text{Cos}\left[\frac{4\pi t}{5}\right]+0.0865095\,\text{Cos}[\pi t]-1.24866\,\text{Sin}\left[\frac{\pi t}{5}\right]+0.312164\,\text{Sin}\left[\frac{2\pi t}{5}\right]+2.45758\times10^{-16}\,\text{Sin}\left[\frac{3\pi t}{5}\right]-0.0780411\,\text{Sin}\left[\frac{4\pi t}{5}\right]+0.0499463\,\text{Sin}[\pi t]\right)$$

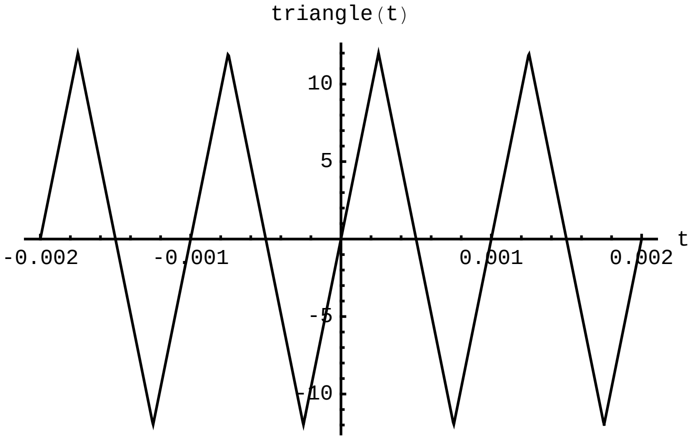
```
0.683918 Cos[0.628319 t] + 0.170979 Cos[1.25664 t] +
 0.0427449 Cos[2.51327 t] + 0.0273567 Cos[3.14159 t] - 0.39486 Sin[0.628319 t] +
 0.0987151 Sin[1.25664 t] - 0.0246788 Sin[2.51327 t] + 0.0157944 Sin[3.14159 t]
```

```
Plot[{triangle[t], triangle2N[t], triangle5N[t]}, {t, -T, T}, AxesLabel →
   {"t", "triangle(t), Fourier: order 2, order 5"}, PlotStyle → {Black, Red, Blue}];
```



- ***Triangle, da A. Poto, "Elettronica", pag. 11, Esercizio 1, corretto***

```
a = 12; T = 10^-3; n = 2; b = T / n;
core[t_] := { a + (-2 a / b) t                                0 ≤ t < T / n
            { (2 a / (T - b)) t - (2 a / (T - b)) (T / n) - a  T / n ≤ t < T
triangle[t_] := core[Mod [t - T / 4, T]]
Plot[Evaluate[triangle[t] ], {t, -2 T, 2 T}, AxesLabel → {"t", "triangle(t)"}];
```




```
<< Calculus`FourierTransform`
```

```
triangle5[t_] =
 FourierTrigSeries[triangle[t], t, 5, FourierParameters -> {0, 1 / T}] // Expand
% //
 N
```

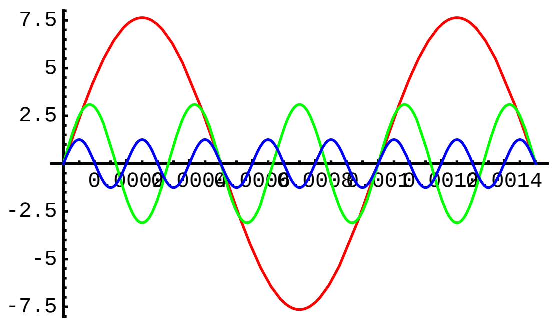
$$\frac{96\,\mathrm{Sin}[2000\,\pi\,t]}{\pi^2} - \frac{32\,\mathrm{Sin}[6000\,\pi\,t]}{3\,\pi^2} + \frac{96\,\mathrm{Sin}[10\,000\,\pi\,t]}{25\,\pi^2}$$

9.72683 Sin[6283.19 t] - 1.08076 Sin[18 849.6 t] + 0.389073 Sin[31 415.9 t]

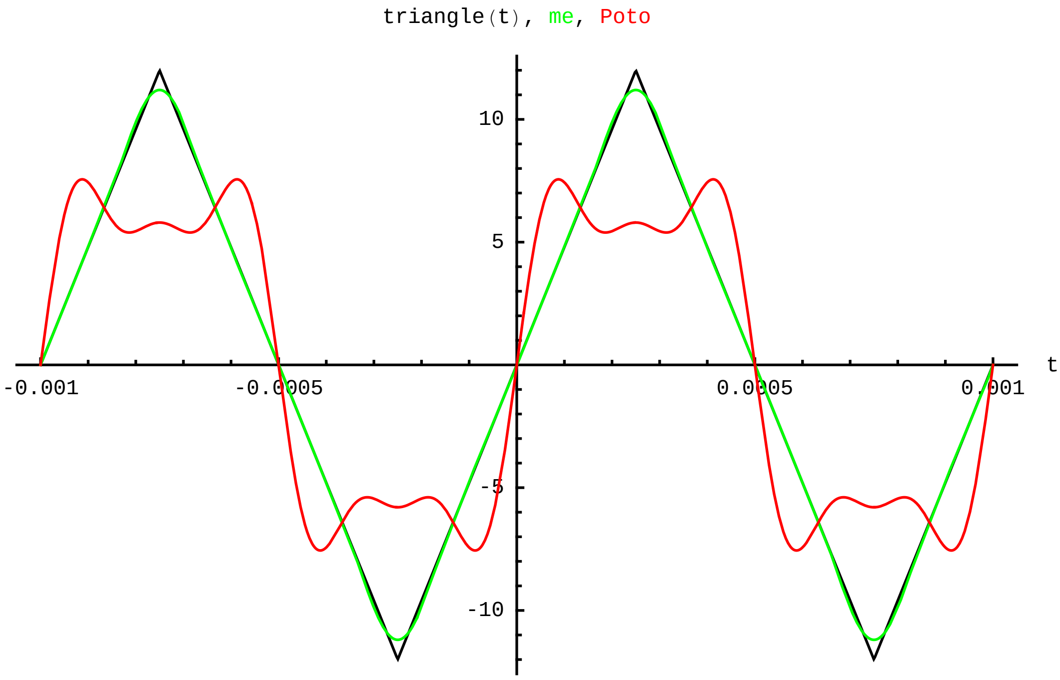
```
PotoT = Table[12 (2 / π)^n Sin[n 2 π 1000 t], {n, 1, 5, 2}];
Poto[t_] = Plus @@ PotoT
% // N
Plot[Evaluate[PotoT], {t, 0, 1.5 T}, PlotStyle → {Hue[0], Hue[1 / 3], Hue[2 / 3]}];
```

$$\frac{24\,\mathrm{Sin}[2000\,\pi\,t]}{\pi} + \frac{96\,\mathrm{Sin}[6000\,\pi\,t]}{\pi^3} + \frac{384\,\mathrm{Sin}[10\,000\,\pi\,t]}{\pi^5}$$

7.63944 Sin[6283.19 t] + 3.09615 Sin[18 849.6 t] + 1.25482 Sin[31 415.9 t]



```
Plot[{triangle[t], triangle5[t], Poto[t]}, {t, -T, T},
  PlotStyle → {Black, Green, Red}, AxesLabel → {"t", "triangle(t), me, Poto"}];
```



Appendice 5.2.A Introduzione al funzionamento di Wolfram Research *Mathematica*

$$\left\{\text{"Hello world!"}, 1+1, \text{Expand}[(a+b)^9], 4\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\right\}$$

$$\{\text{Hello world!}, 2, a^9 + 9a^8 b + 36a^7 b^2 + 84a^6 b^3 + 126a^5 b^4 + 126a^4 b^5 + 84a^3 b^6 + 36a^2 b^7 + 9ab^8 + b^9, \pi\}$$

```
imax = 1; jmax = 8;
Graustufen = Table[GrayLevel[1 - j / (jmax - 1)], {i, 0, imax - 1}, {j, 0, jmax - 1}];
Show[Graphics[RasterArray[Graustufen]],
  AspectRatio → Automatic, Frame → True, FrameTicks → None];
```

Überblick der Definitionen

- **The Fourier transform of a function $f(t)$ is by default defined to be** $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$.
- **The inverse Fourier transform of a function $F(\omega)$ is by default defined as** $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega t}\, d\omega$.

- **Fourier trigonometric series (of *a* periodic function of *t*)**

$$f(t) = |b|^{(1+a)/2}\left(c_0 + \sum_{n=1}^{k} c_n \cos(2\pi b n t) + d_n \sin(2\pi b n t)\right)$$

$$c_0 = |b|^{(1-a)/2} \int_{-1/(2|b|)}^{1/(2|b|)} f(t)\, dt$$

$$c_n = 2\,|b|^{(1-a)/2} \int_{-1/(2|b|)}^{1/(2|b|)} f(t) \cos(2\pi b n t)\, dt$$

$$d_n = 2\,|b|^{(1-a)/2} \int_{-1/(2|b|)}^{1/(2|b|)} f(t) \sin(2\pi b n t)\, dt$$

- **Fourier exponential series (of *a* periodic function of *t*)**

$$f(t) = |b|^{(1+a)/2} \sum_{n=-k}^{k} F_n\, e^{-2\pi i b n t}$$

$$F_n = |b|^{(1-a)/2} \int_{-1/(2|b|)}^{1/(2|b|)} f(t)\, e^{2\pi i b n t}\, dt$$

```
? FourierTrigSeries
```

FourierTrigSeries[expr, t, k] gives the kth order Fourier trigonometric series approximation to the periodic function of t that is equal to expr for -1/2 <= t <= 1/2, and has a period of 1. FourierTrigSeries[expr, t, k, FourierParameters -> {a, b}] gives the kth order Fourier trigonometric series approximation to the periodic function of t that is equal to expr for -1/(2 Abs[b]) <= t <= 1/(2 Abs[b]), and has a period of 1/Abs[b]. More…

```
? FourierSeries
```

FourierSeries[expr, t, k] gives the kth order Fourier exponential series approximation to the periodic function of t that is equal to expr for -1/2 <= t <= 1/2, and has a period of 1. FourierSeries[expr, t, k, FourierParameters -> {a, b}] gives the kth order Fourier exponential series approximation to the periodic function of t that is equal to expr for -1/(2 Abs[b]) <= t <= 1/(2 Abs[b]), and has a period of 1/Abs[b]. More…

```
sincos = A Sin[ω t] + B Cos[ω t]
S = Sqrt[A^2 + B^2]
ϕ = ArcTan[A, B]
sinϕ = S Sin[ω t + ϕ] // TrigToExp // FullSimplify
sinϕ === sincos
```

B Cos[t ω] + A Sin[t ω]

$\sqrt{A^2 + B^2}$

ArcTan[A, B]

B Cos[t ω] + A Sin[t ω]

True

Fourier trigonometric series

```
<< Calculus`FourierTransform`
```

```
FourierCosCoefficient[f[t], t, 0, FourierParameters -> {0, 1 / T}] // TraditionalForm
```

$$\frac{\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}} f(t)\, dt}{\sqrt{|T|}}$$

FourierCosCoefficient[f[t], t, n, FourierParameters -> {0, 1 / T}]

$$\frac{1}{\sqrt{\text{Abs}[T]}}\left(2\,\text{Integrate}\left[\text{Cos}\left[\frac{2\,n\,\pi\,t}{T}\right]f[t],\left\{t,-\frac{\text{Abs}[T]}{2},\frac{\text{Abs}[T]}{2}\right\},\text{Assumptions}\rightarrow\{n\in\text{Integers},n\geq 0\}\right]\right)$$

FourierSinCoefficient[f[t], t, n, FourierParameters -> {0, 1 / T}]

$$\frac{1}{\sqrt{\text{Abs}[T]}}\left(2\,\text{Integrate}\left[f[t]\,\text{Sin}\left[\frac{2\,n\,\pi\,t}{T}\right],\left\{t,-\frac{\text{Abs}[T]}{2},\frac{\text{Abs}[T]}{2}\right\},\text{Assumptions}\rightarrow\{n\in\text{Integers},n\geq 1\}\right]\right)$$

FourierTrigSeries[f[t], t, 3, FourierParameters -> {0, 1 / T}] // Expand // TraditionalForm

$$\frac{\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}} f(t)\,dt}{|T|}+\frac{2\cos\left(\frac{2\pi t}{T}\right)\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}}\cos\left(\frac{2\pi t}{T}\right)f(t)\,dt}{|T|}+\frac{2\cos\left(\frac{4\pi t}{T}\right)\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}}\cos\left(\frac{4\pi t}{T}\right)f(t)\,dt}{|T|}+\frac{2\cos\left(\frac{6\pi t}{T}\right)\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}}\cos\left(\frac{6\pi t}{T}\right)f(t)\,dt}{|T|}+$$
$$\frac{2\left(\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}} f(t)\sin\left(\frac{2\pi t}{T}\right)dt\right)\sin\left(\frac{2\pi t}{T}\right)}{|T|}+\frac{2\left(\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}} f(t)\sin\left(\frac{4\pi t}{T}\right)dt\right)\sin\left(\frac{4\pi t}{T}\right)}{|T|}+\frac{2\left(\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}} f(t)\sin\left(\frac{6\pi t}{T}\right)dt\right)\sin\left(\frac{6\pi t}{T}\right)}{|T|}$$

Fourier exponential series

<< Calculus`FourierTransform`

FourierCoefficient[f[t], t, n, FourierParameters -> {0, 1 / T}]

$$\frac{1}{\sqrt{\text{Abs}[T]}}\,\text{Integrate}\left[e^{\frac{2\,i\,n\,\pi\,t}{T}}f[t],\left\{t,-\frac{\text{Abs}[T]}{2},\frac{\text{Abs}[T]}{2}\right\},\text{Assumptions}\rightarrow\{n\in\text{Integers}\}\right]$$

FourierSeries[f[t], t, 3, FourierParameters -> {0, 1 / T}] // Expand // TraditionalForm

$$\frac{\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}} f(t)\,dt}{|T|}+\frac{e^{\frac{2i\pi t}{T}}\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}} e^{-\frac{2i\pi t}{T}}f(t)\,dt}{|T|}+\frac{e^{-\frac{2i\pi t}{T}}\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}} e^{\frac{2i\pi t}{T}}f(t)\,dt}{|T|}+$$
$$\frac{e^{\frac{4i\pi t}{T}}\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}} e^{-\frac{4i\pi t}{T}}f(t)\,dt}{|T|}+\frac{e^{-\frac{4i\pi t}{T}}\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}} e^{\frac{4i\pi t}{T}}f(t)\,dt}{|T|}+\frac{e^{\frac{6i\pi t}{T}}\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}} e^{-\frac{6i\pi t}{T}}f(t)\,dt}{|T|}+\frac{e^{-\frac{6i\pi t}{T}}\int_{-\frac{|T|}{2}}^{\frac{|T|}{2}} e^{\frac{6i\pi t}{T}}f(t)\,dt}{|T|}$$

"Esisitono 10 tipi di persone: quelli che capiscono il codice binario e quelli che non lo capiscono."